CAPS Match 2024: Solutions and Marking schemes

ISTA, Austria
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Problem 1. Determine whether there exist 2024 distinct positive integers satisfying the following: If we consider every possible ratio between two distinct numbers (we include both a/b and b/a), we will obtain numbers with finite decimal expansions (after the decimal point) of mutually distinct non-zero lengths. (Patrik Bak, Slovakia)

Solution. We will show these numbers exist. For that we define sequences $a_1, a_2, \dots, a_{2024}$ and $b_1, b_2, \dots, b_{2024}$ and then consider numbers $c_i = 2^{a_i} \cdot 5^{b_i}$ for $i = 1, 2, \dots, 2024$.

We choose the sequences a_i and b_i in such a way that a_i is increasing, b_i is decreasing, and the differences $a_i - a_j$ and $b_j - b_i$ were all mutually distinct for all indices $i > j$. This will be enough because

$$\frac{c_i}{c_j} = \frac{2^{a_i} \cdot 5^{b_i}}{2^{a_j} \cdot 5^{b_j}} = \frac{2^{a_i - a_j}}{5^{b_j - b_i}},$$

this number has a decimal expansion of a length $b_j - b_i$, whereas analogously, $\frac{c_j}{c_i}$ has a length of $a_i - a_j$.

We now construct the needed sequences, starting with a_i. We will do it inductively. Take $a_1 = 1$, $a_2 = 2$. When we have the numbers $a_1, a_2, \dots, a_i$, then by choosing $a_{i+1} = 2a_i$ we will achieve $a_{i+1} - a_i > a_i - a_1$, therefore all newly added differences will be higher than the previous ones.

We can construct b_i similarly, starting at the end by taking $b_{2024} = a_{2024}$, then $b_{2023} = 2b_{2024}$, and so on. Since $b_{2023} - b_{2024} = a_{2024}$, all the differences in b_i will be at least $b_{2023} - b_{2024} = a_{2024}$.

Remark: In our construction, $a_i = 2^{i-1}$ and $b_i = 2^{4049-i}$.

Marking scheme.

Partial scores for otherwise incomplete solutions:

- (A_0) Realising that the numbers must be of form $2^{a_i} \cdot 5^{b_i}$ +1 p
- (A_1) Realising that a_i must be increasing and b_i decreasing +1 p
- (A_2) Stating that number of the form $N/5^a$ and $N/2^a$ have decimal expansion of length a. +1 p
- (A_3) Proving that having all differences $a_i - a_j$ and $b_i - b_j$ are mutually distinct is sufficient +2 p
- (A_4) Constructing the sequences a_i and b_i. +2 p

Deductions for essentially complete solutions:

- (C) An essentially complete solution 7p
- (F_0) Missing proof that the sequences a_i and b_i satisfy the condition in A_3. −1 p
- (F_1) Any other minor flaw −1 p

Define $A = \sum_{i=0}^{4} A_i$ and $B = C + F_0 + F_1$. Give $\max\{A, B\}$ points.

Remark: The construction from the official solution is (up to the choice of exponents a_i, b_i and possible multiplication by any integer) unique. Disregard this freedom when marking these solutions.

Problem 2. For a positive integer n, an *n-configuration* is a family of sets $\langle A_{i,j}\rangle_{1\le i,j\le n}$. An n-configuration is called *sweet* if for every pair of indices (i,j) with $1\le i\le n-1$ and $1\le j\le n$ we have $A_{i,j}\subseteq A_{i+1,j}$ and $A_{j,i}\subseteq A_{j,i+1}$. Let $f(n,k)$ denote the number of sweet n-configurations such that $A_{n,n}\subseteq\{1,2,\dots,k\}$. Determine which number is larger: $f(2024,2024^2)$ or $f(2024^2,2024)$. (Wojciech Nadara, Poland)

Solution. Consider a sweet n-configuration $\langle A_{i,j}\rangle_{1\le i,j\le n}$ with $A_{n,n}\subset\{1,2,\dots,k\}$. For any $x\in\{1,2,\dots,k\}$ and $i\in\{1,2,\dots,n\}$ define

$$p_x(i)=|\{j\colon x\in A_{i,j}\}|\,.$$

Since $A_{i,j}\subseteq A_{i,j+1}$ for all suitable i,j, the set $\{j\colon x\in A_{i,j}\}$ consists of $p_x(i)$ largest elements of $\{1,2,\dots,n\}$. Since $A_{i,j}\subseteq A_{i+1,j}$ for all suitable i,j, the function $p_x\colon\{1,2,\dots,n\}\to\{0,1,2,\dots,n\}$ is nondecreasing. Therefore every sweet n-configuration determines a family $\langle p_x\rangle_{1\le x\le k}$ of nondecreasing functions $p_x\colon\{1,2,\dots,n\}\to\{0,1,\dots,n\}$. Conversely, every such a family determines a sweet n-configuration $\langle A_{i,j}\rangle_{1\le i,j\le n}$ with $A_{n,n}\subset\{1,2,\dots,k\}$ in the following way: $A_{i,j}=\{x\in\{1,2,\dots,k\}\colon j\ge n+1-p_x(i)\}$. Therefore $f(n,k)=g(n)^k$ where $g(n)$ is the number of nondecreasing functions $p\colon\{1,2,\dots,n\}\to\{0,1,\dots,n\}$.

Using the stars-and-bars method, there is a bijection between the family of nondecreasing functions $p\colon\{1,2,\dots,n\}\to\{0,1,\dots,n\}$ and the set of sequences consisting of n stars and n bars. The bijection is given by

$$p\longrightarrow \underbrace{**\dots*}_{p(1)}|\underbrace{**\dots*}_{p(2)-p(1)}|\underbrace{**\dots*}_{p(3)-p(2)}|\dots|\underbrace{**\dots*}_{p(n)-p(n-1)}|\underbrace{**\dots*}_{n-p(n)}$$

Thus $g(n)=\binom{2n}{n}$.

The problem boils down to determining which of the numbers

$$\binom{2n}{n}^{n^2},\quad \binom{2n^2}{n^2}^{n},$$

where $n=2024$, is larger. Note that

$$\binom{2n^2}{n^2}=\frac{\prod_{i=1}^{n^2}(n^2+i)}{\prod_{i=1}^{n^2}i}=\prod_{i=1}^{n^2}\left(\frac{n^2+i}{i}\right)=\prod_{j=0}^{n-1}\prod_{i=1}^{n}\frac{n^2+jn+i}{jn+i}>\prod_{j=0}^{n-1}\left(\frac{n^2+jn+n}{jn+n}\right)^n=$$
$$=\prod_{j=0}^{n-1}\left(\frac{n+j+1}{j+1}\right)^n=\left(\prod_{j=1}^{n}\frac{n+j}{j}\right)^n=\binom{2n}{n}^n$$

and therefore

$$\binom{2n^2}{n^2}^n>\binom{2n}{n}^{n^2}.$$

Remark: A sketch of a slightly different way of thinking about f(n,k)$=\binom{2n}{n}^k$: Consider an $n\times n$ table. In a cell with coordinates (i,j), list all the elements of the set $A_{i,j}$. Fix an element $x\in\{1,\dots,k\}$ and consider the cells that contain the number x. By the condition, those cells form a region closed under making a step right and making a step up. Such regions are delimited by grid paths that start at $[0,n]$, end at $[n,0]$, and only steps right or down. There are $\binom{2n}{n}$ possible paths for each x, thus $f(n,k)=\binom{2n}{n}^k$.

Marking scheme. Partial scores for otherwise incomplete solutions:

(A_0)	Guessing the answer	0p
(A_1)	Showing that $f(n,k)=g(n)^k$ for any reasonable definition of g.	+2 p
(A_2)	Showing that $g(n)=\binom{2n}{n}$	+3 p
(A_3)	Showing the inequality	+2 p

Deductions for essentially complete solutions:

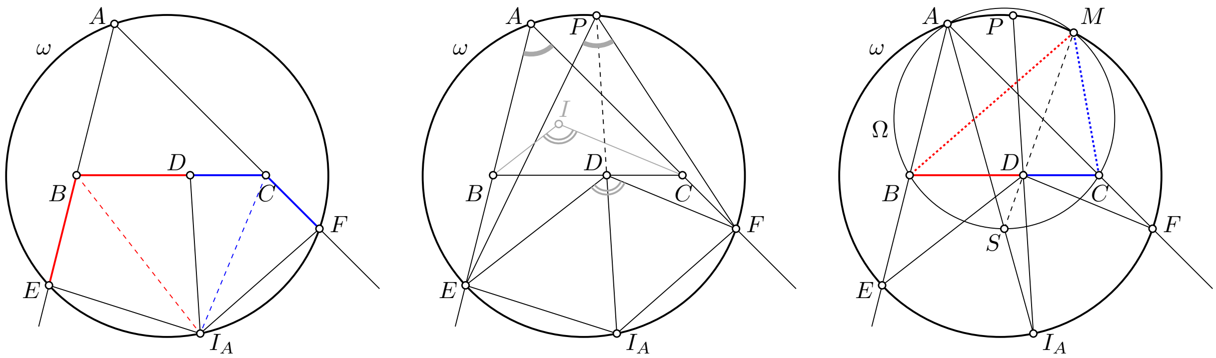
(C)	An essentially complete solution	7p
(F)	Minor flaw	-1p

Define $A = \sum_{i=0}^{3} A_i$ and $B = C + F$. Give $\max\{A, B\}$ points.

Problem 3. Let ABC be a triangle and D a point on its side BC. Points E, F lie on the lines AB, AC beyond vertices B, C, respectively, such that $BE = BD$ and $CF = CD$. Let P be a point such that D is the incenter of triangle PEF. Prove that P lies inside the circumcircle Ω of triangle ABC or on it. (Josef Tkadlec, Czech Republic)

Solution. Let ω the circumcircle of triangle AEF and let I_A be the A-excenter of triangle ABC. First, we prove that I_A is the midpoint of the arc EF of ω that does not contain point A (see the left figure).

To that end, note that since I_A lies on the external angle bisector of $\angle B$ and $BE = BD$, triangles BEI_A and BDI_A are congruent (SAS). Similarly, triangles CFI_A and CDI_A are congruent, so in particular $I_AE = I_AF$. Moreover, $\angle BEI_A + \angle CFI_A = \angle BDI_A + \angle CDI_A = 180$, hence the points A, E, I_A, F lie on a single circle in this order.



Next, we prove that P is the second intersection of I_AD and ω (see the middle figure). Let I be the incenter of triangle ABC. Then $ED \parallel BI$ and $DF \parallel IC$. Setting $\angle EAF = \alpha$, we get $\angle EDF = \angle BIC = 90 + \frac{1}{2}\alpha$, thus $\angle EPF = \alpha = \angle EAF$, so P lies on ω. Since I_A is the midpoint of arc, it lies on the angle bisector PD, so P lies both on I_AD and on ω as claimed.

Finally, we show that P lies on that arc of ω which lies inside Ω (see the right figure). Let $M \neq A$ be the second intersection of ω and Ω (if they are tangent, we set $M = A$). Then M is the center of the spiral similarity that maps BE to CF (alternatively, we angle-chase that triangles MBE and MCF are similar by AA). Thus $MB/MC = BE/CF = BD/DC$, so MD is the angle bisector of BMC, and so it passes through the midpoint S of the arc BC of Ω that does not contain A.

Now forget about points B, C, E, F and focus on circles Ω, ω and on the points A, M, I_A, S, D, P. Circles Ω and ω share points A and M. Being the A-excenter of ABC, point I_A belongs to that arc AM of ω which lies outside of Ω (e.g. since $AI_A > AS$). Point S lies on the segment AI_A and point D lies on the segment SM, so point D lies inside the angle AI_AM. Thus, point $P = I_AD \cap \omega$ belongs to the other arc AM of ω than I_A, namely to the one which lies inside Ω.

Marking scheme. Partial scores for otherwise incomplete solutions:

(A_1) Proving that E, F, P, A lie on a circle. 1 p

(A_2) Proving that I_A is the midpoint of the arc EF in ω. 1 p

(A_3) Introducing M and showing that it is sufficient to show that P lies on the arc AM not containing I_A and that I_A belongs to the arc AM which lies outside Ω. 1 p

(A_4) Proving the similarity of triangles MBE and MCF. 1 p

(A_5) Proving that M, D, S are collinear. 2 p

(A_6) Proving that P lies on the arc AM not containing I_A. 1 p

Deductions for essentially complete solutions:

(C)	Full solution	7 p
(F_0)	Minor mistake (i.e. not showing that I_A lies outside Ω or equivalent)	-1 p

Define $A = \sum_{i=1}^{6} A_i$ and $B = C + F_0$. Give $\max\{A, B\}$ points.

Incomplete computational solutions are not worth points unless they reach any of the points above and the geometrical interpretation is stated.

Problem 4. Let $ABCD$ be a quadrilateral, such that $AB = BC = CD$. There are points X, Y on rays CA, BD, respectively, such that $BX = CY$. Let P, Q, R, S be the midpoints of segments BX, CY, XD, YA, respectively. Prove that points P, Q, R, S lie on a circle. (Michal Pecho, Slovakia)

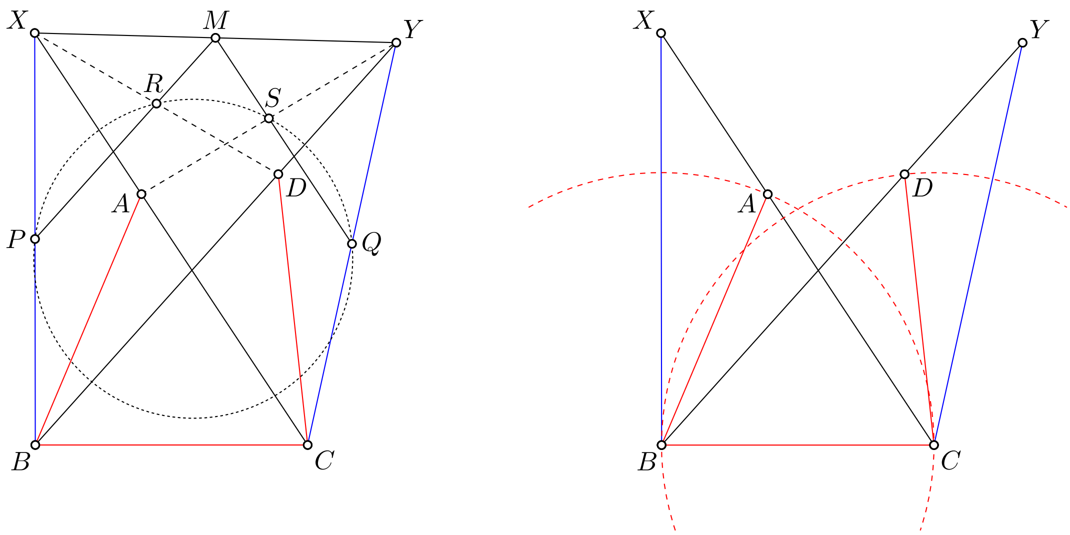
Solution. Let M be the midpoint of XY. Note that PR is midline in triangles XBD and XBY, hence M lies on PR. Analogously M lies on QS.

Let ω_1 be a circle with center B and radius $AB = BC$ and ω_2 be a circle with center C and radius $BC = CD$.

Distance of X from center of ω_1 is the same sa distance of Y from center of ω_2 and also ω_1 and ω_2 have radius of same size, hence power of X with respect to ω_1 is the same as power of Y with respect to ω_2, so

$$XA \cdot XC = YD \cdot YB.$$

Using homotheties centered at X, Y we get that $MS \cdot MQ = MR \cdot MP$ and thus points P, Q, R, S lie on a circle.



Marking scheme. Partial scores for otherwise incomplete solutions:

- (A) Prove that PR and QS intersect in the middle of XY. 2 p
  - (A_1) Hypothesis that PR and QS intersect at XY. 1 p
- (B) Stating that $MR \cdot MP = MS \cdot MQ$ suffices for the problem statement to hold. 0 p
- (C) Prove that it suffices to show $XA \cdot XC = YD \cdot YB$, for instance by reducing (B) to (C) using the mentioned homothety. 2 p
- (D) Prove that $XA \cdot XC = YD \cdot YB$. 3 p
  - (D_0) Having circles with radius BC in a diagram. 0 p
  - (D_1) Verbally considering circle(s) with radius BC as a means of proving (D). 1 p

Deductions for essentially complete solutions:

- (P) Full solution 7 p
- (F_0) Minor mistake -1 p

Define $X = \max\{A, A_1\} + B + C + \max\{D, D_0, D_1\}$ and $Y = P + F_0$. Give $\max\{X, Y\}$ points.

Incomplete computational solutions are not worth points unless they reach any of the points above and the geometrical interpretation is stated.

Problem 5. Let $\alpha \neq 0$ be a real number. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2+y^2) = f(x-y)f(x+y) + \alpha y f(y)$$

holds for all x, $y \in \mathbb{R}$. (Walther Janous, Austria)

Solution. *Answer:* For every $\alpha \neq 0$, the zero function and the function with value 1 at 0, but 0 elsewhere are solutions. For $\alpha = 2$, the identity function $x \mapsto x$ is another solution.

Solution-check. The linear function clearly works. Consider the function f such that $f(0) = 1$ and $f(x) = 0$ otherwise. Note that $yf(y) = 0$ for all real numbers y. Then it is sufficient to realize that $f(x^2+y^2) \neq 0$ iff $x = 0 \wedge y = 0$. Similarly $f(x-y)f(x+y) \neq 0$ iff $x+y = x-y = 0 \Leftrightarrow x = 0 \wedge y = 0$ which shows that also this function is a solution.

Proof. Denote by $P(x,y)$ the proposition in the problem statement. Comparing $P(x,y)$ with $P(x,-y)$ yields $f(y) = -f(-y)$ for all $y \neq 0$. Using this equality, $P(y,x)$ shows that $2f(x-y)f(x+y) = \alpha(xf(x) - yf(y))$ for $x \neq y$. Plugging this into the original equation, we obtain

$$f(x^2+y^2) = \frac{\alpha}{2}(xf(x) + yf(y))$$

for $x \neq y$. Setting $y = 0$ in this equation shows $f(x^2) = \alpha x f(x)/2$ for $x \neq 0$, whereas $P(x,0)$ gives $f(x^2) = f(x)^2$ for all $x \in \mathbb{R}$. Hence $\alpha x f(x)/2 = f(x)^2$, that is, $f(x) = 0$ or $f(x) = \alpha x/2$ for $x \neq 0$. In particular, if $f(x) \neq 0$, then $f(x) = \alpha x/2$. On the other hand, $P(0,0)$ shows $f(0) = f(0)^2$ and therefore $f(0) = 0$ or $f(0) = 1$.

Consider first the case that $f(x) = 0$ for all $x \neq 0$. Then both possible values for $f(0)$ yield functions fulfilling the original equation (if $(x,y) \neq (0,0)$, all terms in $P(x,y)$ are zero anyway and $(x,y) = (0,0)$ was treated before).

Now for the other case: There is a real number $z \neq 0$ satisfying $f(z) = \alpha z/2$. Then $f(z^2) = f(z)^2 = (\alpha/2)^2 z^2 \neq 0$, and hence $f(z^2) = (\alpha/2)z^2$. By comparing the last two statements, we obtain $\alpha = 2$ and then $f(z) = z$.

- $f(0) = 1$. Consider $P(z/2, z/2)$: $f(z^2/2) = z + zf(z/2)$. The left-hand side is 0 or $z^2/2$, the right-hand side z or $z + z^2/2$. Since $z \neq 0$, only $z^2/2 = z \iff z = 2$ and $0 = z + z^2/2 \iff z = -2$ are possible. Either way, $f(2) = 2$ and $f(-2) = -2$, because f is odd. But then $f(4) = f(2^2) = f(2)^2 = 4$, which is impossible, because we just proved that $z = 2$ and $z = -2$ are the only real numbers with $f(z) = z$.
- $f(0) = 0$. We show that $f(x) = 0$ for all positive reals x if f is not the identity function:
  (1) There are $0 < a < b$ with $f(a) = 0$, $f(b) = b$. Then $P(x,y)$ for $x = \sqrt{b-a}$ and $y = \sqrt{a}$ yields
  $$0 \neq b = f(b) = f(x-y)f(x+y) + 2f(a) = f(x-y)f(x+y),$$
  hence $f(x-y) = x-y$ and $f(x+y) = x+y$ and $b = x^2 - y^2 = b - 2a$, forcing the contradiction $a = 0$.
  (2) There are $0 < a < b$ with $f(a) = a$, $f(b) = 0$. Analogous to Case 1, we arrive at the contradiction $b = 0$ when investigating $P(\sqrt{b-a}, \sqrt{a})$.

  Except for the identity, we only have $f(x) = 0$ for $x > 0$ and thus $f(x) = 0$ for $x \neq 0$ as possible solution, which we have already found and treated before.

Marking scheme. Partial scores for otherwise incomplete solutions:

(A_0)	Guessing all the solutions with solution check	1p
(A_1)	Proving that f is odd	1p
(A_2)	Obtaining $f(x) \cdot (f(x) - \alpha x/2) = 0$	3 p

Deductions for essentially complete solutions:

(C)	An essentially complete solution	7p
(F_0)	Mistake in casework which leads to missing a solution	-2 p
(F_1)	Solution-check missing (especially for the non-linear function)	-1 p
(F_2)	Other minor flaw	-1 p

Define $A = \max\{A_0, A_1, A_2\}$ and $B = C + \min\{F_0, F_1\} + F_2$. Give $\max\{A, B\}$ points.

Problem 6. Determine whether there exist infinitely many triples (a, b, c) of positive integers such that p divides $\lfloor (a + b\sqrt{2024})^p \rfloor - c$ for every prime p.

Note: $\lfloor x \rfloor$ denotes the largest integer not larger than x. (Walther Janous, Austria)

Solution.

Let $D := 2024$. Consider any pair of positive integers (a, b) such that $0 < a - b\sqrt{D} < 1$. One can easily find an infinite number of such pairs by choosing $a = \lceil b\sqrt{D} \rceil$. Then

$$(a + b\sqrt{D})^p + (a - b\sqrt{D})^p = 2a^p + 2\sum_{k=1}^{\infty} \binom{p}{2k} a^{p-2k} b^{2k} D^k \in \mathbb{Z}$$

is larger than $(a+b\sqrt{D})^p$, since we add a positive term, but it is smaller than $(a+b\sqrt{D})^p+1$. As it is integer and $p \mid \binom{p}{2k}$ for all $1 \le k \le \frac{p-1}{2}$, we see that

$$\begin{aligned} \left\lfloor (a + b\sqrt{D})^p \right\rfloor &= (a + b\sqrt{D})^p + (a - b\sqrt{D})^p - 1 \\ &= 2a^p + 2\sum_{k=1}^{\infty} \binom{p}{2k} a^{p-2k} b^{2k} D^k - 1 \equiv 2a - 1 \pmod{p} \end{aligned}$$

by Fermat's little theorem. Observe that this congruence is also valid for $p = 2$, although $2 \nmid \binom{2}{2\cdot 1}$, because the sum is taken twice anyway. Therefore, choosing $c := 2a - 1$, we get $p \mid \left\lfloor (a + b\sqrt{D})^p \right\rfloor - c$ for all primes p.

In summary, for any positive integer b we get a triple $(\lceil sqrt2024b \rceil, b, 2\lceil sqrt2024b \rceil - 1)$ that has the desired property.

Marking scheme. Partial scores for otherwise incomplete solutions:

- (A_0) Guessing the answer 0p
- (A_1) Stating the correct construction of numbers a, b (without a proof) 1p
- (B_1) Performing "reasonable" computation with $a + b\sqrt{D}$ and $a - b\sqrt{D}$ 1p
- (B_2) Showing that $(a + b\sqrt{D})^p + (a - b\sqrt{D})^p$ is an integer. 2p
- (B_3) Showing that $(a + b\sqrt{D})^p + (a - b\sqrt{D})^p \equiv 2a \pmod{p}$. 4 p

Deductions for essentially complete solutions:

- (C) An essentially complete solution 7p
- (F_0) Missing and argument that the construction is giving us infinitely many solutions -1p
- (F_1) Other minor flaw (e.g. mistake in computation that leads to wrong construction) -1p

Define $B = \max\{B_1, B_2, B_3\}$ and $E = C + F_0 + F_1$. Give $\max\{A_1 + B, E\}$ points.